International Journal of Theoretical Physics, Vol. 29, No. 1, 1990

Metric-Torsion Gauge Theory of Continuum Line Defects

A. Verçin[1]

Received August 11, 1989

Basic points underlying the geometrization of continuum defects are discussed. Following an analogy with gravitational gauge theories, a metric-torsion gauge theory of continuum line defects is developed. Gauge-invariant action integrals are constructed and their equations of motion are obtained. A Lagrangian containing curvature terms up to second power has constant-curvature solutions. In linear approximation these solutions correspond to line defects which form closed loops separately.

1. INTRODUCTION

The geometrization of a physical theory, i.e., expressing it in a differential geometric form, requires the following (Schultz, 1985):

(i) Identification of differential geometric concepts with certain physically measurable quantities.

(ii) Specification of how the metric, curvature, and torsion of the space corresponding to the underlying continuum are generated or determined by the physical objects of the theory.

(iii) Specification of how physical objects (particles, defects, etc.) behave in this mathematical space.

Up to now, in the continuum mechanics of defects much valuable work concerning the first and the second points has been done (Kröner, 1981; Kléman, 1983; and references given there), but the third point needs further study.

The basic geometric identifications made in the continuum mechanics of defects are that:

(i) The underlying continuum used for the description of physical phenomena related to defects is a differentiable manifold (body manifold).

[1]Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey.

(ii) The dislocation and disclination line densities are identified with the torsion and curvature of that manifold, respectively, and the metric is given by $g_{ij} = \delta_{ij} - 2e_{ij}$, where the Kronecker symbols δ_{ij} and e_{ij} are the metric components of the defect-free body manifold and the components of strain tensor, respectively.

These identifications are valid only in three dimensions, i.e., for the statics of defects.

Following a close analogy with gravitational gauge theories, this paper investigates the third point stated above. After discussing the motivation for the study and giving the basic differential geometric notation used in the subsequent analyses, the differential geometric structure of the body manifold corresponding to various situations and a Lagrangian formulation of the program is given. The investigation is based on the two identifications stated above, and will be carried out in arbitrary dimension, to open the way to the dynamics of defects. Problems of this approach and some points for further investigation are mentioned.

2. MOTIVATION AND NOTATION

Gauge models of defects in (dis)ordered media have been formulated for various fields of condensed matter physics, such as crystals and amorphous solids (Dzyaloshinskii, 1981; Rivier and Duffy, 1982; Kadic and Edelen, 1983; Venkataraman and Sahoo, 1985; 1986; Kleinert, 1987; Trzesowski, 1987). At intermediate length scales these discrete systems are described within the continuum approach, which is the approach used in this paper. Such a continuum theory of defects describes well only those phenomena which can be classified as large-scale phenomena. The gauge models of defects mentioned above are formulated in analogy with classical Yang–Mills theories. Perhaps it is not equally well known that gravitational theories, including the Einstein theory, are gauge theories of a different kind.

By a classical gauge theory is meant any physical theory which includes among its dynamical variables a connection on a principal G-bundle over a (flat or curved) base manifold M. The structure group G is a Lie group often called a gauge group. A connection 1-form Γ on Π: $P \to M$ describes a gauge configuration and a local section s: $U \to P$, $U \subset M$, $\Pi \circ s = Id_M$, defines a gauge, where P is the principal G-bundle, Π is the projection map, and Id_M is an identity transformation of M. The pullback of the connection $\omega = s^*\Gamma$ and curvature $R = s^*\Omega$ are called, respectively, the potential and field strength of the gauge configuration in the gauge s, where Ω is the covariant differentiation of Γ, $\Omega = D\Gamma$ (Trautman, 1984).

The most important difference between Yang–Mills-type gauge models and gravitational gauge models is that the underlying bundle of the latter is the bundle of linear frames $L(M)$, which is a collection of all linear frames defined at each point of M. The structure group of $L(M)$ is $GL(m)$, the general linear group in m dimensions. $L(M)$ is "concrete" and has more structure than the "abstract" bundles of the other gauge theories. The additional structure is due to the soldering form θ, which upon covariant differentiation leads to torsion, $\Theta = D\theta$. Since the vector bundle involved is the "most natural" bundle associated with a manifold, gravitational gauge theories are the "most natural" gauge theories (Bergmann and Flaherty, 1978; Trautman, 1982).

By making use of the pullback map s^*, the geometric structure of the bundle can be mapped on M itself, that is, Cartan structure equations (we use the Einstein summation convention)

$$R^i{}_j = D\omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \Lambda \omega^k{}_j = \tfrac{1}{2} R^i{}_{jkl} e^k \Lambda e^l \tag{1a}$$

$$T^i = De^i = de^i + \omega^i{}_j \Lambda e^j = \tfrac{1}{2} T^i{}_{kl} e^k \Lambda e^l \tag{1b}$$

and their integrability conditions (Bianchi identities)

$$DR^i{}_j = 0, \qquad DT^i = R^i{}_j \Lambda e^j \tag{2}$$

are also satisfied on M, where $R^i{}_j = s^*\Omega^i{}_j$, $T^i = s^*\Theta^i$, $\omega^i{}_j = s^*\Gamma^i{}_j$, $e^i = s^*\theta^i$ (d and Λ denote the exterior derivative and exterior multiplication, respectively). Under a change of local section associated with a gauge transformation $(a^i{}_j) \in GL(m)$, m being the dimension of M, base and connection 1-forms change according to

$$e^i = a^i{}_j \acute{e}^j, \qquad a^i{}_k \acute{\omega}^k{}_j = \omega^i{}_k a^k{}_j + da^i{}_j \tag{3}$$

The corresponding changes of curvature and torsion 2-forms are $a^i{}_k \acute{R}^k{}_j = R^i{}_k a^k{}_j$ and $T^i = a^i{}_k \acute{T}^k$, respectively (Kobayashi and Nomizu, 1963).

So far, there is no relation between connection and metric, i.e., they are taken as two independent fields describing the geometric structure. But, among all connections, the metric compatible connection that satisfies the metricity condition

$$Dg_{ij} = dg_{ij} - g_{ik}\omega^k{}_j - g_{jk}\omega^k{}_i = 0 \tag{4}$$

is the "most physically reasonable" one. First, condition (4) means that parallel displacement, which is an operation to compare geometric objects defined at two distinct points of M, is an isometry. For example, the angle between two parallelly displaced vectors and their lengths remains unchanged. Second, condition (4) allows us to have locally Euclidean structure at each point of M. This is important, to know the degree of deviation from a flat structure at a point. Third, it will be clear in the following investigation that the metricity condition is the main geometric property of a continuum surviving after a plastic deformation. As emphasized by Hehl *et al.* (1976), "the metricity postulate is an *a posteriori* constraint which reflects in a precise manner the results of numerous experiments." I assume condition (4), which implies $R_{ij} = -R_{ji}$, throughout this work. Thus, the geometric structure can be described by g and ω, or equivalently, by g and torsion T. Because of condition (4), the number of independent field variables is

$$\frac{m(m+1)}{2} + \frac{m^2(m-1)}{2} = \frac{m(m^2+1)}{2}$$

The linear connection can be written as

$$\omega^i{}_j = \tilde{\omega}^i{}_j + \tau^i{}_j \tag{5}$$

where $\tilde{\omega}^i{}_j$ is the torsion-free Levi-Civita connection determined through the equation $de^i + \tilde{\omega}^i{}_j \Lambda e^j = 0$, and $\tau^i{}_j$, which represents the non-Riemannian part of $\omega^i{}_j$, is called the contorsion 1-form. While the antisymmetric part of $\tau^i{}_j$ specifies torsion through the equation $T^i = \tau^i{}_j \Lambda e^j$, its symmetric part is important in the specification of parallelly transported vector fields. In local (x^i) coordinates, writing

$$\omega^i{}_j = \Gamma^i{}_{kj}\, dx^k, \qquad \tilde{\omega}^i{}_j = \{{}^{\,i}_{k\,j}\}\, dx^k$$

$$\tau^i{}_j = \tau^i{}_{jk}\, dx^k, \qquad T^i = \tfrac{1}{2} T^i{}_{jk}\, dx^j \Lambda\, dx^k \qquad (T^i{}_{jk} = -T^i{}_{kj})$$

and using condition (4) and formula (5), we have

$$\Gamma^i{}_{kj} = \{{}^{\,i}_{k\,k}\} + \tau^i{}_{jk} \tag{6}$$

where

$$\{{}^{\,i}_{k\,j}\} = \{{}^{\,i}_{j\,k}\} = \tfrac{1}{2} g^{il}(g_{lk,j} + g_{lj,k} - g_{jk,l}) \tag{7a}$$

$$\tau^i{}_{jk} = -\tfrac{1}{2}(T^i{}_{jk} + T_{jk}{}^i + T_{kj}{}^i) \tag{7b}$$

(I use the abbreviation $\partial g_{ij}/\partial x^k = g_{ij,k}$). Working in an orthonormal basis $e^\alpha = f^\alpha_i\, dx^i$ is easier and often more helpful when one looks for an operational interpretation of quantities appearing in the formalism (henceforth, Greek indices are used to denote noncoordinate indices). Since in an orthonormal basis $g_{\alpha\beta} = \delta_{\alpha\beta}$, condition (4) gives $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, and by using formula (1b), we obtain

$$\Gamma_{\alpha\lambda\beta} = \tau_{\alpha\beta\lambda} + \tfrac{1}{2}(f_{\alpha\beta\lambda} + f_{\beta\lambda\alpha} + f_{\lambda\beta\alpha}) \tag{8}$$

where $f^\alpha{}_{\beta\lambda} = f^i_\beta f^j_\lambda (f^\alpha_{i,j} - f^\alpha_{j,i})$ is called the object of anholonomity, which measures the noncommutativity of the orthonormal basis. f^i_α and f^α_i are reciprocals of each other, $f^i{}_\alpha f^\alpha_j = \delta^i{}_j$, $f^\alpha_i f^i_\beta = \delta^\alpha{}_\beta$. Now that parallel transport of an orthonormal basis is given by $de^\alpha = -\omega^\alpha{}_\beta \Lambda e^\beta$, $\omega^\alpha{}_\beta$ represents the rotation of a parallelly transported orthonormal basis relative to the given one. As is obvious from equation (8), this rotation consists of two parts: one the Ricci rotation due to the object of anholonomity, and one due to contorsion called the "added twist." That is, in an orthonormal basis, instead of the metric, the base 1-forms play the role of independent degrees of freedom.

Finally, the Ricci 1-form R_α, which is the only essential contraction of the curvature 2-form, and the curvature scalar Q are defined as

$$R_\alpha = i_\alpha R^\alpha{}_\beta, \qquad Q = i_\alpha R^\alpha = i_\alpha i_\beta R^{\beta\alpha} \tag{9}$$

where i_α denotes inner product operation with respect to e_α, $i_\alpha e^\beta = \delta^\beta{}_\alpha$, and e_α is the dual basis of e^α. Symmetrization and antisymmetrization are denoted by $(\cdot)$ and $[\cdot]$, respectively, and ε_{ijk} stands for the totally antisymmetric unit tensor with $\varepsilon_{123} = 1$.

3. DEFECTS IN LINEAR ELASTICITY THEORY

The only quantity that determines the physically observable internal stresses due to defects is the symmetric, divergence-free incompatibility tensor

$$\eta_{ij} = \varepsilon_i{}^{km} \varepsilon_j{}^{ln} e_{mn,kl} \tag{10}$$

(Kröner, 1981; Kosevich, 1980; Kléman, 1980). In the case of small deformations, which is the standard approximation of the linear elasticity theory, $e_{mn} = \beta_{(mn)}$, β_{mn} being the deformation field. Dislocation and disclination line densities are defined to be

$$\alpha_{ij} = \varepsilon_i{}^{kl} \beta_{lj,k} \tag{11a}$$

$$\theta_{ij} = \varepsilon_j{}^{ln} \varepsilon_i{}^{km} e_{mn,kl} - \varepsilon_i{}^{kl} (\alpha_{jl} - \tfrac{1}{2} \delta_{jl} \alpha^n{}_n)_{,k} \tag{11b}$$

respectively. (u_n) is the displacement field, $e_{mn} = u_{(n,m)}$ if and only if $\eta = 0$ (St. Venant theorem). Given α_{ij} and θ_{ij}, definitions (11a) and (11b) are differential equations for unknown e_{mn}. In that case their integrability conditions (kinematic equations of defects) are

$$\theta^i{}_{j,i} = 0, \qquad \alpha^k{}_{m,k} = -\varepsilon_m{}^{ij} \theta_{ij} \tag{12}$$

These equations express the fact that disclination lines cannot end within body, and dislocation lines can end on disclination lines. Now using equations (10) and (11), η can be written in terms of defects as

$$\eta_{ij} = \theta_{(ij)} + \varepsilon_i{}^{km} \alpha_{jm,k} + \varepsilon_j{}^{km} \alpha_{im,k} \tag{13}$$

This form of η clearly shows the contribution of defects to the incompatibility, which is a measure of the deviation from the defect-free case.

The basic mathematical problem of the theory of internal stresses consists in calculating stress and strain in the body, given the incompatibility tensor η, i.e., given the defects. The basic equations to be solved are the force and moment equilibrium conditions $\sigma^{i}{}_{j,i}=0$, $\xi^{i}{}_{j,i}=-\varepsilon_{j}{}^{km}\sigma_{km}$ and the constitutive equations of the media (σ_{ij} and ξ_{ij} denote stress and moment stress tensors, respectively).

The above brief review of the theory of defects is the static case. Currents of defects and interactions of defects are not considered. All the equations considered above are local in the sense that they are not covariant under coordinate changes, i.e., they are coordinate dependent. Their generalization will be given within the differential geometric description of continuum defects.

4. DIFFERENTIAL GEOMETRY OF A CONTINUUM CONTAINING LINE DEFECTS

In continuum mechanics a material body is a three-dimensional differentiable manifold M such that there exist global orientation-preserving diffeomorphisms $\kappa: M \to U \subset E^3$, where U is a connected subset of the three-dimensional Euclidean point space E^3. We call $(\kappa, \kappa(M))$ a configuration of M. In order to avoid problems related to the boundary of M, U will be taken as an open subset of E^3. Let $(\kappa, \kappa(M))$ be a certain distinguished configuration, called a reference configuration, and let (x^A) be a coordinate system on it. By definition, $(\kappa, \kappa(M))$ can be covered by such a single coordinate system. A deformation of the body with respect to $(\kappa, \kappa(M))$ is a mapping of the following form:

$$\lambda = \psi \circ \kappa^{-1}: \quad \kappa(M) \to \Psi(M) \tag{14}$$

$(\lambda, \psi(M))$ is called a current configuration with respect to $(\kappa, \kappa(M))$. The ordered one-parameter family of configuration Ψ_t, $t \in I$, is called a motion of the body, $I=[0, 1]$ being a closed, unit time interval. Two cases concerning mapping (14) are important: (i) a defect-free current configuration, for which λ is a diffeomorphism, and (ii) a current configuration with defects, for which λ is not a diffeomorphism. According to these definitions, defects are obstructions to the existence of diffeomorphisms that uniquely characterize current configurations.

If $(\lambda, \Psi(M))$ is a defect-free current configuration with respect to $(\kappa, \kappa(M))$ then there exists a coordinate system (x^i) on $\psi(M)$ such that $x^i = x^i(x^A)$ and $x^A = x^A(x^i)$ are well-behaved, single-valued, and differentiable functions of their arguments. In that case the matrix F whose entries are $F^{i}{}_{A} = \partial x^i / \partial x^A$ is called the deformation matrix of M with respect to

$(\kappa, \kappa(M))$ and since deformations are orientation preserving, the determinant of F is positive. For the sake of simplicity, I assume that $e_A = \partial/\partial x^A$ is a global orthonormal vector basis such that the metric and connection of $(\kappa, \kappa(M))$ are $g_{AB} = e_A \cdot e_B = \delta_{AB}$ and $\omega^A{}_B = \Gamma^A{}_{CB}\, dx^C = 0$. By using transformation formulas for the metric and connection, the metric and connection of $(\lambda, \Psi(M))$ are

$$g_{ij} = S^A{}_i S^B{}_j \delta_{AB}, \qquad \omega^i{}_j = F^i{}_A\, dS^A{}_j \tag{15}$$

or, in matrix form $g = {}^t SgS$ and $\omega = F\,dS$, where S is in the inverse matrix of F, $SF = FS = 1$, and the superscript t denotes the transpose operation. By applying d to g and ω, one obtains $R = 0$ and $dg = {}^t(g\omega) + (g\omega)$, i.e., the connection of $(\lambda, \Psi(M))$ is metric compatible and flat. As a result, a defect-free current configuration is characterized by a global coordinate basis $e^i = dx^i$ and metric compatible flat connection as

$$e^i = F^i{}_A\, dx^A, \qquad \omega^i{}_j = F^i{}_A\, dS^A{}_j \tag{16}$$

If these equations are regarded as a set of differential equations for (x^i) and $(\omega^i{}_j)$, their integrability conditions are

$$de^i + \omega^i{}_j \Lambda e^j = 0, \qquad d\omega^i{}_j + \omega^i{}_k \Lambda \omega^k{}_j = 0$$
$$(T^i = 0) \qquad\qquad (R^i{}_j = 0)$$

That is, a global coordinate system (x^i) and a metric compatible flat connection can be defined on all of $\Psi(M)$ if and only if the torsion and curvature are zero at each point of $\Psi(M)$. In brief, the torsion and curvature of a current configuration must represent, in one way or another, defects that are obstructions to the existence of diffeomorphisms from a reference configuration to a current configuration.

If the deformation λ is not a diffeomorphism, there is not a coordinate system in the current configuration which can be connected to $(\kappa, \kappa(M))$ by $dx^i = F^i{}_A\, dx^A$. Although one can write $e^\alpha = F^\alpha{}_A\, dx^A$, $F^\alpha{}_A$ is no longer a gradient field, i.e., e^α is a noncoordinate basis. First of all, let us suppose that in local (x^i) coordinates of $(\lambda, \Psi(M))$ the connection is still $\omega^i{}_j = F^i{}_A\, dS^A{}_j$. Thus, $(\lambda, \lambda(M))$ will have $R^i{}_j = 0$ and nonvanishing torsion in the form $T^i = \omega^i{}_j \Lambda\, dx^j$, $DT^i = 0$. Since $T^i = 0$ is the integrability condition of $e^i = F^i{}_A\, dx^A$, in the case of a dislocated configuration ($T^i \neq 0$) the coordinates are anholonomic. As a result, a dislocated configuration has a teleparallel geometric structure characterized by $Dg = 0$, $R^i{}_j = 0$, $T^i \neq 0$.

Locally, teleparallel geometry can be described by a vanishing connection (Kopczyński, 1982). In that case torsion corresponds to the object of anholonomity, $T^\alpha = de^\alpha$, where e^α is an orthonormal basis. In general, a global orthonormal basis may not exist and we may not be able to cover the dislocated configuration by local coordinates such that in their intersection $\omega^i{}_j = 0$ is always satisfied.

If the connection is taken to be a first-order infinitesimal quantity (linear approximation), $DT^i = dT^i = 0$ gives $T^i = \omega^i{}_j \Lambda\, dx^i = d\beta^i$. From $\omega^i{}_j = \Gamma^i{}_{kj}\, dx^k$ and $\beta^i = \beta^i{}_j\, dx^j = w^i{}_j\, dx^j + e^i{}_j\, dx^j$ one obtains

$$de_{ij} = \tfrac{1}{2}(\Gamma_{ikj} + \Gamma_{jki})\, dx^k, \qquad dw_{ij} = \tfrac{1}{2}(\Gamma_{ikj} - \Gamma_{jki})\, dx^k$$

where w_{ij} is the antisymmetric part of the distorsion field β_{ij} (in linear elasticity theory). These equations give a local interpretation of the connection in continuum mechanics.

If the only defects present are disclinations, i.e., $R^i{}_j \neq 0$, $T^i = 0$, the connection of $(\lambda, \Psi(M))$ cannot be the flat $\omega^i{}_j = F^i{}_A\, dS^A{}_j$ connection, but it is the Levi-Civita connection (7a) uniquely determined from the metricity condition (4). Thus, disclinated current configurations have a Riemannian geometric structure characterized by $Dg = 0$, $R^i{}_j = D\omega^i{}_j$ and $DR^i{}_j = 0$.

Finally, if both kinds of defects are present, the corresponding geometric structure will be a non-Riemannian geometry characterized, in addition to the metricity condition, by the Cartan structure equations (1a) and (1b) and the Bianchi identities (2). In linear approximation, the Bianchi identities $dT^i = R^i{}_j \Lambda e^j$ and $dR^i{}_j = 0$ are nothing more than the kinematic equations of defects given by equations (12), provided that the identifications

$$\alpha^{ij} = \varepsilon^{ikm} T^j{}_{km}, \qquad \theta^{ij} = \tfrac{1}{2} \varepsilon^{imn} \varepsilon^{jkl} R_{klmn} \tag{17}$$

are accepted. Thus, the Cartan structure equations and the Bianchi identities are natural covariant generalizations of the definition of defects and the kinematic equations of defects, respectively. Consistent with this fact, covariant generalization of the other equations must be given via a minimal replacement argument ("comma goes to semicolon" rule); for example, $\sigma^i{}_{j,i} = 0$ must be replaced by $\sigma^i{}_{j;i} = \sigma^i{}_{j,i} + \Gamma^i{}_{ik} \sigma^k{}_j - \Gamma^k{}_{ji} \sigma^i{}_k = 0$.

Special cases of current configurations and corresponding geometric structures are shown in Table I. The hierarchical order of structures is represented schematically by the following diagram:

$$\begin{array}{c} \text{Metric-affine} \\ \text{geometry} \end{array} \xrightarrow{Dg=0} \begin{array}{c} \text{Riemann-Cartan} \\ \text{geometry} \end{array} \left\{ \begin{array}{c} \xrightarrow{T=0} \begin{array}{c} \text{Riemannian} \\ \text{geometry} \end{array} \xrightarrow{R=0} \\ \xrightarrow{R=0} \begin{array}{c} \text{Teleparallel} \\ \text{geometry} \end{array} \xrightarrow{T=0} \end{array} \right\} \begin{array}{c} \text{Euclidean} \\ \text{geometry} \end{array}$$

An important point about this diagram that must be emphasized is that the conditions given before a geometry are to be taken as constraints on that geometry (Trautman, 1982; Kopczyński, 1982).

Now, the basic mathematical problem of the internal stresses due to line defects has turned out to be a purely geometric problem: find the metric and connection for a given torsion and curvature. Because of the covariant

Table I. Differential Geometric Structures Corresponding to a Continuum Containing Line Defects

Defect-free configuration $\alpha^{ij}=0, \quad \theta^{ij}=0$	Euclidean geometry $T^i=0, \quad R^i{}_j=0$
Dislocated configuration $\alpha^i{}_{j,i}=0, \quad \theta^{ij}=0$	Teleparallel geometry $T^i=de^i, \quad dT^i=0, \quad R^i{}_j=0$ $T^i=De^i, \quad DT^i=0, \quad R^i{}_j=0$
Disclinated configuration $\alpha^{ij}=0, \quad \theta^i{}_{j,i}=0$	Riemannian geometry $T^i=0, \quad DR^i{}_j=0$
Dislocated and disclinated configuration $\alpha^i{}_{j,i}=\varepsilon_j{}^{km}\theta_{km}, \quad \theta^i{}_{j,i}=0$	Riemann-Cartan geometry $DT^i=R^i{}_j\Lambda e^j, \quad DR^i{}_j=0$

generalization of the "old equations," the connection has appeared as a new field variable. Note that in this form of the problem the torsion and curvature must be given in order to solve the metric and connection. On the other hand, the main purpose of the gauge theories of defects is to postulate some field equations of defects, i.e., some differential equations for the torsion and curvature. These equations can be written in terms of the metric and connection by using the Cartan structure equations. Following an analogy with gravitational gauge theories, the field equations of defects will be given in the next section.

5. VARIATIONAL PRINCIPLE FOR THE METRIC-TORSION GAUGE THEORY OF THE CONTINUUM LINE DEFECTS

In this section I formulate a variational principle which will give field equations of the continuum line defects. In this formulation, I will follow a close analogy with gravitational gauge theories. To open the way to the dynamics of defects, the analysis will be carried out in a dimension-independent manner, and will be modern in the sense that all calculations will be performed in terms of differential forms.

Let us model a continuum containing line defects by an action integral over the body manifold M; $I=\int_M L$. The Lagrangian L is an m-form (m is the dimension of M) and it is a function of the field variables g_{ij}, ω_{ij}, and their exterior derivatives, i.e., $L=L(g_{ij}, \omega_{ij}, dg_{ij}, d\omega_{ij})$. The statics and the dynamics of defects correspond to $m=3$ and $m=4$, respectively. To facilitate the calculations, orthonormal bases will be used exclusively. In an orthonormal basis e^α, the metric has the form $g_{\alpha\beta}=\delta_{\alpha\beta}$ and because of the metricity condition (4), connection 1-forms are antisymmetric, $\omega_{\alpha\beta}=-\omega_{\beta\alpha}$. Since in an orthonormal basis the metric is fixed, the variation of L

with respect to the metric is equivalently accomplished by variation with respect to the orthonormal basis 1-forms e^α. Thus, without any loss of generality, we write $L=L(e^\alpha, \omega^\alpha{}_\beta, de^\alpha, d\omega^\alpha{}_\beta)$, or, by using the Cartan structure equations, $L=L(e^\alpha, \omega^\alpha{}_\beta, R^\alpha{}_\beta, T^\alpha)$. Then, the left variation of L with respect to the field variables is defined as

$$\delta L = \delta e^\alpha \wedge \frac{\partial L}{\partial e^\alpha} + \tfrac{1}{2}\delta\omega^\alpha{}_\beta \wedge \frac{\partial L}{\partial \omega^\alpha{}_\beta} + \tfrac{1}{2}\delta R^\alpha{}_\beta \wedge \frac{\partial L}{\partial R^\alpha{}_\beta} + \delta T^\alpha \wedge \frac{\partial L}{\partial T^\alpha}$$

where $\partial L/\partial\omega^\alpha{}_\beta$ and $\partial L/\partial R^\alpha{}_\beta$ are antisymmetric by definition. Equations (1a) and (1b) enable us to write ∂L as

$$\delta L = -\delta e^\alpha \wedge E_\alpha + \tfrac{1}{2}\delta\omega^\alpha{}_\beta \wedge C^\beta{}_\alpha + d\left(\delta e^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + \tfrac{1}{2}\delta\omega^\alpha{}_\beta \wedge \frac{\partial L}{\partial R^\alpha{}_\beta}\right) \quad (18)$$

where the Einstein $(m-1)$-form E_α and the Cartan $(m-1)$-form $C^\beta{}_\alpha$ are

$$E_\alpha = -\frac{\partial L}{\partial e^\alpha} - D\left(\frac{\partial L}{\partial T^\alpha}\right) \quad (19a)$$

$$C^\beta{}_\alpha = \frac{\partial L}{\partial \omega^\alpha{}_\beta} + 2e^\beta \wedge \frac{\partial L}{\partial T^\alpha} + D\left(\frac{\partial L}{\partial R^\alpha{}_\beta}\right) \qquad (C_{\alpha\beta} + C_{\beta\alpha} = 0) \quad (19b)$$

I will call the equations of motion resulting from $\partial I = \int_M \partial L = 0$, i.e., the equations

$$E_\alpha = 0, \qquad C^\beta{}_\alpha = 0, \qquad \alpha, \beta = 1, 2, \ldots, m \quad (20)$$

the field equations of the continuum line defects. Since these equations are $(m-1)$-form equations, their number is $m^2 + m[m(m-1)/2] = m^2(m+1)/2$. However, they are not all independent; two basic requirements must be satisfied. First, to ensure the independence of equations (20) from the choice of the orthonormal basis e^α, L must be invariant under $SO(3)$ transformations. Second, L must be invariant under diffeomorphic transformations of M, since otherwise L would be position dependent. The second requirement means that once line defects are produced by a plastic deformation, they must be irremovable by an elastic deformation.

These two requirements reduce the number of equations (20) from $m^2(m+1)/2$ to $m(m^2-1)/2$. To see this, let us take the variations of e^α and $\omega^\alpha{}_\beta$ under an infinitesimal $SO(3)$ transformation $a^\alpha{}_\beta(x) = (\delta^\alpha{}_\beta + \varepsilon^\alpha{}_\beta)$. Using equations (3), we have $\delta e^\alpha = \acute{e}^\alpha - e^\alpha = -\varepsilon^\alpha{}_\beta e^\beta$ and $\delta\omega^\alpha{}_\beta = D\varepsilon^\alpha{}_\beta$ with $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. Substitution of δe^α and $\delta\omega^\alpha{}_\beta$ into formula (18) leads to

$$\delta L = L' - L = \varepsilon^\alpha{}_\beta (e^\beta \wedge E_\alpha - \tfrac{1}{2} DC^\alpha{}_\beta) + \tfrac{1}{2} d\left(\varepsilon^\alpha{}_\beta \frac{\partial L}{\partial \omega^\alpha{}_\beta}\right)$$

Since $\varepsilon_{[\alpha\beta]}$ and $d\varepsilon_{[\alpha\beta]}$ can be taken to be arbitrary at each point, $\delta L = 0$ gives

$$\frac{\partial L}{\partial \omega^\alpha{}_\beta} = 0, \qquad DC_{\alpha\beta} = e_\alpha \wedge E_\beta - e_\beta \wedge E_\alpha \quad (21)$$

The first equation means that L cannot contain the connection explicitly. The second one expresses the fact that the field equations of defects are not all independent and their number is reduced by $m(m-1)/2$.

To ensure the second requirement, it is enough to take a one-parameter family of diffeomorphism Ψ_t and use the Lie derivative with respect to the vector field X generated by Ψ_t. In that case the following equations are obtained:

$$DE_\alpha = T^\beta{}_{\alpha\lambda} e^\lambda \Lambda E_\beta - \tfrac{1}{2} R^\beta{}_{\lambda\alpha\eta} e^\eta \Lambda C^\lambda{}_\beta \tag{22a}$$

$$\frac{\partial L}{\partial e^\alpha} + T^\beta{}_{\alpha\lambda} e^\lambda \Lambda \frac{\partial L}{\partial T^\beta} + \tfrac{1}{2} R^\beta{}_{\lambda\alpha\eta} e^\eta \Lambda \frac{\partial L}{\partial R^\beta{}_\lambda} - L_\alpha = 0 \tag{22b}$$

where $i_X L = X^\alpha L_\alpha$ (Kopczyński, 1982). Equation (22b) is nothing more than an equivalent expression of E_α, and equation (22a) gives m independent constraints on the possible field equations. So, equations (21) and (22a) reduce the number of equations (20) from $m^2(m+1)/2$ to $m(m^2-1)/2$. On the other hand, the number of independent field variables is $m(m^2+1)/2$. As a result, a complete specification of the geometric structure of the continuum needs m more equations. In other words, there remain m degrees of freedom to be gauged. This freedom is called gauge freedom and it is specified by a gauge condition.

Keeping an analogy with the three best-known classical gauge theories (Maxwell's electrodynamics, Einstein's theory of general relativity, and Yang–Mills theory), only gauge-invariant Lagrangians containing terms up to second powers of curvature and torsion will be investigated. So, following this analogy, consideration will be restricted to the following Lagrangians:

$$L_1(R) = 2L_{\mathrm{HE}} = R_{\alpha\beta} \Lambda {}^*(e^\alpha \Lambda e^\beta) \tag{23a}$$

$$L_2(R) = f(Q)^*1 \tag{23b}$$

$$L_3(R) = R_{\alpha\beta} \Lambda {}^* R^{\alpha\beta} \tag{23c}$$

$$L_1(T) = T_\alpha \Lambda {}^* T^\alpha \tag{23d}$$

$$L_2(T) = (e_\alpha \Lambda T_\beta) \Lambda {}^*(e^\beta \Lambda T^\alpha) \tag{23e}$$

$$L_3(T) = (e_\alpha \Lambda T^\alpha) \Lambda {}^*(e_\beta \Lambda T^\beta) \tag{23f}$$

$$L_4(T) = (i_\alpha T^\alpha) \Lambda {}^*(i_\beta T^\beta) \tag{23g}$$

$$L(P) = P_\alpha \Lambda P^\alpha \tag{23h}$$

$$L(T, P) = T_\alpha \Lambda P^\alpha \tag{23i}$$

where L_{HE} is the well-known Hilbert–Einstein Lagrangian and * is the Hodge duality mapping. It must be noted that $L(T, P)$ is possible only in three dimensions.

We have found the formula

$$\delta(\omega\Lambda{}^*\beta)=\delta\omega\Lambda{}^*\beta+\delta\beta\Lambda{}^*\omega-\delta e^\alpha\Lambda[(i_\alpha\omega)\Lambda{}^*\beta+(-1)^{p+1}\omega\Lambda i_\alpha{}^*\beta] \quad (24)$$

which simplifies the variational calculations, where ω and β are two arbitrary p-forms. For $\omega=\beta$ we have

$$\delta(\beta\Lambda{}^*\beta)=2\delta\beta\Lambda{}^*\beta-\delta e^\alpha\Lambda\kappa_\alpha(\beta) \quad (25)$$

where

$$\kappa_\alpha(\beta)=(i_\alpha\beta)\Lambda{}^*\beta+(-1)^{p+1}\beta\Lambda(i_\alpha{}^*\beta)$$

is called the stress form. Using formulas (24) and (25), the variations of the Lagrangians listed above are calculated, and the resulting field equations of the continuum line defects are given together in Table II. In this table the abbreviations

$$\sigma=i_\alpha T^\alpha, \qquad Z^\alpha=e^\alpha\Lambda\sigma+T^\alpha, \qquad \xi=e_\alpha\Lambda T^\alpha$$

are used. The explicit expressions of the stress forms are

$$\kappa_\alpha(R)=(i_\alpha R_{\beta\gamma})\Lambda{}^*R^{\beta\gamma}-R_{\beta\gamma}\Lambda i_\alpha{}^*R^{\beta\gamma}$$

$$\kappa_\alpha(T)=(i_\alpha T^\beta)\Lambda{}^*T_\beta-T^\beta\Lambda i_\alpha{}^*T_\beta$$

$$\kappa_\alpha(T\Lambda e)=[i_\alpha(T^\beta\Lambda e^\gamma)]\Lambda{}^*(T_\gamma\Lambda e_\beta)+T^\beta\Lambda e^\gamma\Lambda i_\alpha{}^*(T_\gamma\Lambda e_\beta)$$

$$\kappa_\alpha(\xi)=(i_\alpha\xi)\Lambda{}^*\xi+\xi\Lambda i_\alpha{}^*\xi$$

Since $L_4(T)=L_1(T)-L_2(T)$ and in three dimensions $2L(P)=L_3(R)+\frac{1}{2}Q^{2*}1$, field equations corresponding to $L_4(T)$ and $L(P)$ are not included in Table II.

Table II gives the field equations of the continuum line defects described by the Lagrangians listed above. I claim that each of these Lagrangians, or a linear combination of them, models a special continuum containing line defects. To see this, let us investigate a special linear combination of $L_2(R)$ and $L_3(R)$ in the form of

$$L=\tfrac{1}{2}R_{\alpha\beta}\Lambda{}^*R^{\alpha\beta}+f(Q){}^*1 \quad (26)$$

Table II. Field Equations of the Continuum Line Defects Obtained from $\delta I=\int_M \delta L=0$[a]

L	$E_\alpha=0$	$\frac{1}{2}C^{\alpha\beta}=0$
$L_1(R)$	$R_{\gamma\beta}\Lambda{}^*(e^\gamma\Lambda e^\beta\Lambda e_\alpha)=0$	$D^*(e^\alpha\Lambda e^\beta)=0$
$L_2(R)$	$f'R_{\gamma\beta}\Lambda{}^*(e^\gamma\Lambda e^\beta\Lambda e_\alpha)-(Qf'-f){}^*e_\alpha=0$	${}^*(e^\alpha\Lambda e^\beta)\Lambda\, df'+f'D^*(e^\alpha\Lambda e^\beta)=0$
$L_3(R)$	$-\kappa_\alpha(R)=0$	$2D^*R^{\alpha\beta}=0$
$L_1(T)$	$-[\kappa_\alpha(T)-2D^*T_\alpha]=0$	$2e^{[\beta}\Lambda{}^*T^{\alpha]}=0$
$L_2(T)$	$-[\kappa_\alpha(T\Lambda e)-2D^*Z_\alpha-2T^\beta\Lambda{}^*(e_\beta\Lambda T_\alpha)]=0$	$2e^{[\beta}\Lambda{}^*Z^{\alpha]}=0$
$L_3(T)$	$-[\kappa_\alpha(\xi)-2T_\alpha\Lambda{}^*\xi+2e_\alpha\Lambda D^*\xi]=0$	$-2e^\alpha\Lambda e^\beta\Lambda{}^*\xi=0$
$L(T,P)$	$DP_\alpha-i_\beta(T^\gamma\Lambda R^\beta{}_{\gamma\alpha\eta}e^\eta)=0$	$e^{[\beta}\Lambda P^{\alpha]}-Di^{[\alpha}T^{\beta]}=0$

[a] $f'=df(Q)/dQ$, $e^{[\beta}\Lambda{}^*T^{\alpha]}=\frac{1}{2}(e^\beta\Lambda{}^*T^\alpha-e^\alpha\Lambda{}^*T^\alpha)$.

where we take $f(Q)=c_1Q+c_2$, c_1 and c_2 are constants, and $*1$ is the m-dimensional volume element. Making use of Table II gives the corresponding field equations:

$$D^*R^{\alpha\beta}=-c_1D^*(e^\alpha\Lambda e^\beta) \tag{27a}$$

$$c_1R_{\alpha\beta}\Lambda^*(e^\alpha\Lambda e^\beta\Lambda e_\gamma)+c_2{}^*e_\gamma=\tfrac{1}{2}\kappa_\gamma(R) \tag{27b}$$

The formal similarity of equation (27a) with the Yang–Mills field equations is obvious. However, equation (27b) has no counterpart in those models. Since in three dimensions $D^*(e^\alpha\Lambda e^\beta)=\varepsilon^{\alpha\beta}{}_\gamma T^\gamma$, it is interesting to note that ${}^*T^\gamma$ is playing the role of a source for curvature in equation (27a) (Dereli and Verçin, 1987; Verçin, 1988).

For $c_1\neq 0$ and $c_2=\frac{1}{2}m(m-1)c_1^2$, equations (27a) and (27b) have a curvature solution as

$$R_{\alpha\beta}=-c_1e_\alpha\Lambda e_\beta \tag{28}$$

This solution represents a space of constant curvature, which plays an important role in the general theory of relativity. The metric and the connection of such a space are well known (Thirring, 1979).

The Bianchi identities and the solution (28) show that the torsion satisfies the equation $DT^\alpha=0$, or it is zero. Thus, the Lagrangian

$$L=\tfrac{1}{2}R_{\alpha\beta}\Lambda^*R^{\alpha\beta}+\left[c_1Q+\frac{m(m-1)}{2}c_1^2\right]*1 \tag{29}$$

has the solutions

$$\text{(i)}\quad R_{\alpha\beta}=-c_1e_\alpha\Lambda e_\beta,\qquad DR_{\alpha\beta}=0,\qquad T_\alpha=0 \tag{30a}$$

$$\text{(ii)}\quad R_{\alpha\beta}=-c_1e_\alpha\Lambda e_\beta,\qquad DR_{\alpha\beta}=0,\qquad T_\alpha\neq 0,\qquad DT_\alpha=0 \tag{30b}$$

In three dimensions, the solution (30a) describes a continuum containing only disclinations. The second solution corresponds to a continuum containing both kinds of defects, which, in the linear approximation, form closed loops separately, without ending on each other.

6. CONCLUSION

In this paper the basic points underlying the geometrization of the continuum defects are discussed and a gauge model for them is formulated in analogy with gauge theories of gravitation rather than the Yang–Mills gauge theories of high-energy physics. The metric and metric compatible connection are the basic field variables of the metric-torsion gauge theory. Gauge-invariant Lagrangians containing torsion and curvature terms up to the second power are constructed and their equations of motion are obtained.

For the sake of clarity, a special Lagrangian is considered and corresponding field equations are compared with the Yang–Mills-type gauge models. It has been shown that the solutions of these equations correspond to a continuum containing line defects, which, in linear approximation, can be interpreted as forming closed loops separately.

It is possible to study the other field equations, or some linear combinations of them. Moreover, one can construct some gauge-invariant Lagrangians containing higher order terms of the torsion and curvature. But, at this level, this will not contribute to a better understanding of the present approach.

Since the formulation is carried out in terms of differential forms, most of the equations are written in an arbitrary dimension. This gives the opportunity to study the dynamics of defects. At this point we encounter two difficulties: (i) the identification of the space-time components of torsion and curvature, and (ii) the construction of a four-dimensional metric. Although the identification of the mixed components with defect currents seems the most plausible solution, this point needs to be discussed further. Concerning the second difficulty, Günther (1983) and Leinkauf (1989) deserve credit for opening the study of a complete geometrization. In their work the speed of sound, which enters a wave equation obtained from the defect equations, is taken as a fundamental kinematic constant like the speed of light in the theory of relativity. The four-dimensional non-Riemannian space so constructed is called "sound space time." On the other hand, this approach has some problems, such as the existence of more than one sound velocity. When these difficulties are overcome, the present formulation can be extended easily for the dynamics of defects.

ACKNOWLEDGMENT

This work was supported in part by the Scientific and Technical Research Council of Turkey (TUBITAK).

REFERENCES

Bergmann, P. G., and Flaherty, E. J. (1978). *Journal of Mathematical Physics*, **19**, 212.

Dereli, T., and Verçin, A. (1987). *Philosophical Magazine*, **56**, 625.

Dzyaloshinskii, I. (1981). In *Physics of Defects* (R. Balian *et al.*, eds.), North-Holland, Amsterdam, p. 320.

Günther, H. (1983). *Annalen der Physik*, **40**, 291.

Hehl, F. W., von der Heyde, P., and Kerlick, G. D. (1976). *Review of Modern Physics*, **8**, 393.

Kadic, A., and Edelen, D. G. B. (1983). *Lecture Notes in Physics*, Vol. 174, Springer-Verlag, Berlin.

Kleinert, H. (1987). *Annalen der Physik*, **44**, 117.

Kléman, M. (1980). In *Dislocations in Solids* (F. R. N. Nabarro, ed.), North-Holland, Amsterdam, p. 244.

Kléman, M. (1983). In *Bifurcation Theory, Mechanics and Physics* (C. P. Bruter *et al.*, eds.), D. Reidel, Dordrecht, p. 331.

Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. 1, Wiley Interscience, New York.

Kopczyński, W. (1982). *Journal of Physics A*, **15**, 493.

Kosevich, A. M. (1980). In *Dislocations in Solids* (F. R. N. Nabarro, ed.), North-Holland, Amsterdam, p. 33.

Kröner, E. (1981). In *Physics of Defects* (R. Balian *et al.*, eds.), North-Holland, Amsterdam, p. 219.

Leinkauf, H. (1989). *Annalen der Physik*, **46**, 15.

Rivier, N., and Duffy, D. M. (1982). *Journal de Physique* **43**, 293.

Schultz, B. F. (1985). *General Relativity*, Cambridge University Press, Cambridge.

Thirring, W. (1979). *A Course in Mathematical Physics*, Vol. 2, Springer-Verlag, Berlin.

Trautman, A. (1982). In *Geometric Techniques in Gauge Theories*, Springer-Verlag, Berlin, p. 179.

Trautman, A. (1984). *Differential Geometry for Physicists*, Bibliopolis, Naples.

Trzesowski, A. (1987). *International Journal of Theoretical Physics*, **26**, 1059.

Venkataraman, G., and Sahoo, D. (1985). *Contemporary Physics*, **26**, 579.

Venkataraman, G., and Sahoo, D. (1986). *Contemporary Physics*, **27**, 3.

Verçin, A. (1988). Ph.D. dissertation, Ankara University, Ankara.